Statistics of the occupation time for a class of Gaussian Markov processes

2001 J. Phys. A: Math. Gen. 34 1247

(http://iopscience.iop.org/0305-4470/34/7/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 1247–1269 www.iop.org/Journals/ja PII: S0305-4470(01)17899-6

Statistics of the occupation time for a class of Gaussian Markov processes

G De Smedt[1], **C Godrèche**[2] **and J M Luck**[1]

[1] Service de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette Cedex, France
[2] Service de Physique de l'État Condensé, CEA Saclay, 91191 Gif-sur-Yvette Cedex, France

E-mail: desmedt@spht.saclay.cea.fr, godreche@spec.saclay.cea.fr and luck@spht.saclay.cea.fr

Received 4 October 2000

Abstract
We revisit the work of Dhar and Majumdar (1999 *Phys. Rev.* E **59** 6413) on the limiting distribution of the temporal mean $M_t = t^{-1}\int_0^t \mathrm{d}u\,\mathrm{sign}\,y_u$, for a Gaussian Markovian process y_t depending on a parameter α, which can be interpreted as Brownian motion in the time scale $t' = t^{2\alpha}$. This quantity, the mean 'magnetization', is simply related to the occupation time of the process, that is the length of time spent on one side of the origin up to time t. Using the fact that the intervals between sign changes of the process form a renewal process on the time scale t', we determine recursively the moments of the mean magnetization. We also find an integral equation for the distribution of M_t. This allows a local analysis of this distribution in the persistence region ($M_t \to \pm 1$), as well as its asymptotic analysis in the regime where α is large. Finally, we put the results thus found in perspective with those obtained by Dhar and Majumdar by another method, based on a formalism due to Kac.

PACS numbers: 0250E, 0250G, 0540

1. Introduction

Consider the stochastic process y_t defined by the Langevin equation

$$\frac{\mathrm{d}y_t}{\mathrm{d}t} = \sqrt{2\alpha}\, t^{\alpha-1/2}\eta_t \qquad (1.1)$$

where α is a positive parameter, and η_t is a Gaussian white noise such that $\langle \eta_t \rangle = 0$ and $\langle \eta_{t_1}\eta_{t_2} \rangle = \delta(t_2 - t_1)$. In the new time scale

$$t' = t^{2\alpha}$$

this process satisfies the usual Langevin equation for one-dimensional Brownian motion,

$$\frac{\mathrm{d}y_{t'}}{\mathrm{d}t'} = \zeta_{t'}$$

where $\zeta_{t'}$ is still a Gaussian white noise, with $\langle \zeta_{t'} \rangle = 0$ and $\langle \zeta_{t'_1} \zeta_{t'_2} \rangle = \delta(t'_2 - t'_1)$. The process defined by (1.1) is a simple example of subordinated Brownian motion [1]. As for Brownian motion itself, it is Gaussian, Markovian and non-stationary.

This process appears in various situations of physical interest. For instance, it is described in [2] as a Markovian approximation to fractional Brownian motion. It also appears in [3] for the special case $\alpha = \frac{1}{4}$, as describing the time evolution of the total magnetization of a Glauber chain undergoing phase ordering.

Dhar and Majumdar [4] raised the question of computing the distribution of the occupation time of this process, that is the length of time spent by the process on one side of the origin up to time t,

$$T_t^{\pm} = \int_0^t \mathrm{d}u \, \frac{1 \pm \sigma_u}{2} \tag{1.2}$$

where $\sigma_t = \operatorname{sign} y_t$, or equivalently of

$$M_t = \frac{1}{t} \int_0^t \mathrm{d}u \, \sigma_u \tag{1.3}$$

where M_t, the temporal mean of σ_t, is hereafter referred to as the 'mean magnetization' by analogy with physical situations where σ_t represents a spin. The distribution of the occupation time bears information on the statistics of persistent events of the process beyond that contained in the persistence exponent [5–11]. This exponent governs the decay $\sim t^{-\theta}$ of the survival probability of the process, that is the probability that the process did not cross the origin up to time t. Actually, for the present case, the determination of θ is trivial, as shown by simple reasoning [4]: the probability for the Brownian process $y_{t'}$ not to change sign up to time t' is known to decay as $(t')^{-1/2}$, hence for the original process it decays as $t^{-\alpha}$. This shows that $\theta = \alpha$.

When $\alpha = \frac{1}{2}$, the distribution of the fraction of time spent on one side of the origin by a random walker, or by Brownian motion, is given, in the long-time regime, by the arcsine law [1, 12]. In contrast, when $\alpha \neq \frac{1}{2}$, the explicit determination of this distribution, or equivalently of the distribution of M_t, seems very difficult. However, as shown in [4], in the long-time regime, the computation of the asymptotic moments $\langle M_t^k \rangle$ can be done recursively, using two different methods, yielding the same results. The first method relies on a formalism due to Kac [13], while the second one originates from [5].

The method used in [5] can be applied to any (smooth) process for which the intervals of time between sign changes are independent, when taken on a logarithmic scale, with finite (i.e. non-zero) mean $\bar{\ell}$. It eventually leads to a recursive determination of the moments of M_t, as $t \to \infty$ (see equation (3.9) below).

Dhar and Majumdar make the observation that, since relations (3.9) are independent of $\bar{\ell}$, they can be applied to the determination of the moments of M_t for the process (1.1). Comparing the resulting expressions of the moments thus obtained to those derived by their alternative method shows that this is indeed the case.

However, it is not obvious to understand why relations (3.9) hold for the (non-smooth) process (1.1), since the assumptions made in order to derive them do not hold for such a process. In particular, while, for the class of models with independent time intervals on a logarithmic scale, and finite $\bar{\ell}$ (for which the method of [5] has been devised), it is natural to work in a logarithmic time scale, since the mean number of sign changes between 0 and t scales as $\langle N_t \rangle \approx (\ln t)/\bar{\ell}$, this is not so in the present case, since $\bar{\ell}$ vanishes asymptotically, and the mean number of sign changes scales as $\langle N_t \rangle \approx 2\pi^{-1/2} t^{\alpha}$ [11]. The validity of relations (3.9) for the process (1.1) therefore requires an explanation.

In this paper, we revisit and extend the study done in [4].

We first give a new derivation of the asymptotic expressions of the moments $\langle M_t^k \rangle$. We start from the same premise as in [5], and then follow another route—more adapted to the process under study—because of the difficulties encountered in applying the step-by-step method of [5] to the present case (sections 2-5). We then identify the symmetry properties of the distributions of the random variables that appear in the computations, and derive a functional integral equation, the solution of which yields the distribution of M_t (section 6). This approach is first checked on the case $\alpha = \frac{1}{2}$ (section 7). It is then successively applied to the study of the local behaviour of this distribution in the persistence region, for general α (section 8), and to the large-α regime (section 9).

We finally discuss some aspects of [4]. We explain why a formal application of the method of [5] to the present case is only heuristic, and give a new interpretation of the results obtained in [4] with the method of Kac, in light of the present work (section 10).

2. Observables of interest

Changes of sign of the process y_t (or zero crossings) occur at discrete instants of time $t_1, t_2, \ldots, t_n, \ldots,$ once the process is suitably regularized at short times. We assume that the process starts at the origin, so that $t_0 = 0$ is also a sign change. Let N_t be the number of sign changes which occurred between 0 and t, i.e. N_t is the random variable for the largest n for which $t_n \leqslant t$.

In the scale t', where the process is (regularized) Brownian motion, sign changes occur at the instants of time[3] $t'_n = (t_n)^{2\alpha}$, and $N_{t'} \equiv N_t$ is the random variable for the largest n for which $t'_n \leqslant t'$. The intervals of time between sign changes are denoted by $\tau'_n = t'_n - t'_{n-1}$. These are independent, identically distributed random variables, with a probability density function $\rho(\tau')$. For large values of τ', $\rho(\tau')$ decays proportionally to $(\tau')^{-3/2}$. This behaviour is independent of the regularizing procedure, while its prefactor just reflects the choice of time units. The density $\rho(\tau')$ is therefore in the basin of attraction of a Lévy law of index $\frac{1}{2}$. We choose units so that we have in Laplace space

$$\mathop{\mathcal{L}}_{\tau'} \rho(\tau') = \hat{\rho}(s) = \left\langle \mathrm{e}^{-s\tau'} \right\rangle \underset{s\to 0}{\approx} 1 - \sqrt{s}. \tag{2.1}$$

The process formed by the independent intervals of time $\tau'_1, \tau'_2, \ldots,$ is known as a renewal process. In the original scale t, the intervals of time $\tau_n = t_n - t_{n-1}$ are *not* independent.

We denote by t_N the instant of the last change of sign of the process before time t. This random variable depends implicitly on time t through N_t. In the scale t', we have $t'_N = (t_N)^{2\alpha}$.

The occupation times T_t^+ and T_t^- (see equation (1.2)) are the lengths of time spent by the sign process σ_t in the + and − states, respectively, up to time t, hence $t = T_t^+ + T_t^-$. They are simply related to the mean magnetization (1.3) by

$$tM_t = T_t^+ - T_t^- = 2T_t^+ - t = t - 2T_t^-.$$

Assume that $y_t > 0$ at $t = 0^+$, i.e. $\sigma_{t=0} = +1$. Then

$$tM_t = \begin{cases} -(t - t_N) + (t_N - t_{N-1}) - \cdots & \text{if} \quad N_t = 2k+1 \text{ (i.e. } \sigma_t = -1) \\ (t - t_N) - (t_N - t_{N-1}) + \cdots & \text{if} \quad N_t = 2k \text{ (i.e. } \sigma_t = +1). \end{cases}$$

The converse holds if $\sigma_{t=0} = -1$. Hence we have, with equal probabilities,

$$M_t = \pm(1 - 2\xi_t) \tag{2.2}$$

[3] Hereafter we denote by a prime any temporal variable in this scale.

where

$$\xi_t = \frac{1}{t}(t_N - t_{N-1} + \cdots)$$

is the fraction of time spent in the state + if $\sigma_t = -1$, and conversely. The latter formula can be rewritten as

$$\xi_t = \frac{t_N}{t} X_N \tag{2.3}$$

where the X_N obey the recursion

$$X_N = 1 - \frac{t_{N-1}}{t_N} X_{N-1} \tag{2.4}$$

with $X_1 = 1$. Both random variables X_N and t_{N-1}/t_N depend implicitly on time t through N_t.

For instance, if $\sigma_{t=0} = +1$ and $N_t = 4$, then

$$\begin{aligned} M_t &= \frac{1}{t}\big((t - t_4) - (t_4 - t_3) + (t_3 - t_2) - (t_2 - t_1) + t_1\big) \\ &= 1 - 2\xi_t \end{aligned}$$

where $\xi_t = (t_4 - t_3 + t_2 - t_1)/t = t_4 X_4/t$, with

$$X_4 = 1 - \frac{t_3}{t_4}\left(1 - \frac{t_2}{t_3}\left(1 - \frac{t_1}{t_2}\right)\right).$$

3. Methods of solution

Equations (2.2)–(2.4) contain in essence the solution to the problem posed, namely the determination of the limiting distribution of the mean magnetization M_t for $t \to \infty$. Unfortunately, no explicit solution can be attained in general.

However, from (2.2)–(2.4), one can obtain recursively the moments of M_t, in the long-time limit. This can be done either along the lines of [5], as done in [4], or by the method of this paper. In this section, we explain the difficulty encountered when applying the method of [5] to the process (1.1), in order to justify the more lengthy path we have adopted for the derivation of the moments. We shall return to the comparison between the two methods in section 10.

3.1. General framework

Assume that, in the long-time regime, the dimensionless random variables t_N/t, t_{N-1}/t_N, X_N, ξ_t, and M_t possess a limiting joint distribution. Define

$$H = \lim_{t\to\infty} \frac{t_N}{t} \qquad F = \lim_{t\to\infty} \frac{t_{N-1}}{t_N}$$
$$X = \lim_{t\to\infty} X_N \qquad \xi = \lim_{t\to\infty} \xi_t \qquad M = \lim_{t\to\infty} M_t.$$

Then the equations to be solved are:

$$X = 1 - FX \tag{3.1}$$
$$\xi = HX \tag{3.2}$$
$$M = \pm(1 - 2\xi). \tag{3.3}$$

These equalities hold in distribution, and the random variables entering them are not independent *a priori*. Equation (3.1) is to be understood as the fixed-point equation corresponding to the recursion (2.4), while (3.2) and (3.3) correspond to (2.3) and (2.2), respectively.

Assume that the distribution of the random variable F is given, and that F is independent of X. Even so, solving (3.1) is difficult in general [14–17]. However, obtaining the moments of X recursively is easier. If, furthermore, H and X are independent and the moments of H are known, then (3.2) and (3.3) determine the moments of M.

3.2. The diffusion equation: a reminder

Such a situation arises precisely in the example treated in [5]: the process y_t is the diffusing field at a fixed point of space, evolving from random initial conditions, and the so-called independent-interval approximation is used [18, 19]. In the long-time regime, the process is stationary in the logarithmic time scale $T = \ln t$. Consequently, the autocorrelation function of the sign process, $A(|\Delta T|) = \langle \sigma_T \sigma_{T+\Delta T} \rangle$, only depends on the difference of logarithmic times [18, 19].

Consider the intervals of time ℓ_N between successive sign changes of the process in the logarithmic time scale, $\ell_N = T_N - T_{N-1}$, or

$$\mathrm{e}^{-\ell_N} = \frac{t_{N-1}}{t_N}. \tag{3.4}$$

The independent-interval approximation consists in considering the intervals ℓ_N as independent, and thus defining a renewal process. The distribution of the random variable ℓ_N can then be derived, in Laplace space, from knowledge of the correlation function $A(|\Delta T|)$. This distribution is found to be independent of time, because the process is stationary in logarithmic time. Its average, $\langle \ell_N \rangle = \bar{\ell}$, is some time-independent positive number. Explicitly,

$$\hat{f}_{\ell_N}(s) = \left\langle \mathrm{e}^{-s\ell_N} \right\rangle = \left\langle F^s \right\rangle = \frac{1 - \bar{\ell} g(s)}{1 + \bar{\ell} g(s)} \tag{3.5}$$

with

$$g(s) = \tfrac{1}{2} s \big(1 - s\hat{A}(s)\big) \tag{3.6}$$

where $\hat{A}(s)$ is the Laplace transform of $A(T)$. In particular, the moments

$$f_k = \left\langle F^k \right\rangle = \left\langle \left(\frac{t_{N-1}}{t_N} \right)^k \right\rangle = \left\langle \mathrm{e}^{-k\ell_N} \right\rangle = \hat{f}_{\ell_N}(k) \tag{3.7}$$

are independent of time. Thus from (3.1) the moments of X are determined recursively in terms of the f_k (see (A.18)).

In the long-time regime, the distribution of the backward recurrence time of the process in the logarithmic scale, $\lambda = T - T_N$, is also independent of time. This logarithmic recurrence time is related to the random variable H by

$$\mathrm{e}^{-\lambda} = \frac{t_N}{t} = H.$$

Its distribution in Laplace space reads

$$\hat{f}_\lambda(s) = \left\langle \mathrm{e}^{-s\lambda} \right\rangle = \left\langle H^s \right\rangle = \frac{2g(s)}{s\left(1 + \bar{\ell} g(s)\right)}. \tag{3.8}$$

The random variables X and H (or λ) are independent. Hence (3.2) and (3.3) determine the moments of M. A remarkable fact is that the moments thus obtained, which are functions of the f_k and of $\bar{\ell}$, become independent of $\bar{\ell}$ when the f_k are expressed in terms of the $\hat{A}(k) \equiv \hat{A}_k$, using equations (3.5)–(3.7). Thus [5]

$$\langle M^2 \rangle = \hat{A}_1$$
$$\langle M^4 \rangle = 1 - \frac{\big(1 - 3\hat{A}_1 + 4\hat{A}_2\big)\big(1 - 3\hat{A}_3\big)}{1 - 2\hat{A}_2} \tag{3.9}$$

and so on.

More generally, the method used in [5] can be applied to any process for which the intervals of time between sign changes are independent, when taken on a logarithmic scale. It eventually leads to a recursive determination of the moments of M, resulting in (3.9).

3.3. The case of the process (1.1)

The situation for the process (1.1) is more difficult because, in the limit $t \to \infty$, $t_{N-1}/t_N \to F = 1$. Hence (3.1) no longer determines X, and furthermore $\langle \ell_N \rangle \to 0$. Now, for the class of models with independent time intervals ℓ_N on a logarithmic scale, and finite (i.e. non-zero) $\langle \ell_N \rangle = \bar{\ell}$, for which the method of [5], sketched above, has been devised, the mean number of sign changes between 0 and t scales as $\langle N_t \rangle \approx (\ln t)/\bar{\ell}$. So, in contrast, in the present case there is no obvious reason to work with this logarithmic time scale, since $\langle \ell_N \rangle$ vanishes asymptotically, and the mean number of sign changes scales as $\langle N_t \rangle \approx 2\pi^{-1/2} t^{\alpha}$ [11].

On the other hand, if time is kept finite, then the time intervals ℓ_N are not independent and the process (1.1) is not stationary, again precluding the application of the method of [5].

A way out of this is to formally apply this method to the process (1.1), without paying attention to the difficulties mentioned above, and taking advantage of the fact that the moments $\langle M^k \rangle$, given by (3.9), are independent of $\bar{\ell}$, and therefore (hopefully) insensitive to the fact that $\langle \ell_N \rangle \to 0$. This approach, which is the one followed by Dhar and Majumdar [4], is, however, only heuristic, as discussed further in section 10.

Our approach relies instead on the fact that the time intervals τ'_n between two sign changes of the process (1.1) form a renewal process (the τ'_n are independent, identically distributed random variables with density $\rho(\tau')$, given by (2.1) for large τ'). This is a fundamental property of the process (1.1), and, in particular, of Brownian motion if $\alpha = \frac{1}{2}$.

This property allows us to determine the limiting distribution f_H of H when $t \to \infty$, and the moments $f_{k,t}$ of the random variable t_{N-1}/t_N, which are now time dependent. We also find the explicit time-dependent expression of $\langle \ell_N \rangle$. Using the original equation (2.4), instead of (3.1), and equations (3.2) and (3.3), we eventually recover the expressions (3.9) of the $\langle M^k \rangle$, thus extending their range of applicability.

We then establish an integral equation for f_X, and study its consequences.

4. Distribution of t_N/t

Using the independence of the $\tau'_1, \tau'_2, \ldots$, we first determine the distribution of the random variable t'_N, from which we then deduce that of t_N. The method used below is borrowed from [11], where a thorough study of the statistics of the occupation time of renewal processes can be found.

We denote by $f_{t'_N,N}$ the joint probability density of the random variables t'_N and N_t. It reads

$$f_{t'_N,N}(t'; y, n) = \frac{\mathrm{d}}{\mathrm{d}y}\mathcal{P}\left(t'_N < y, N_{t'} = n\right)$$
$$= \left\langle \delta(y - t'_N) I(t'_n < t' < t'_{n+1}) \right\rangle$$

where $I(t'_n < t' < t'_{n+1}) = 1$ if the event inside the parentheses occurs, and 0 if not. The brackets denote the average over $\tau'_1, \tau'_2, \ldots$. Summing over n gives the distribution of t'_N,

$$f_{t'_N}(t'; y) = \sum_{n=0}^{\infty} f_{t'_N,N}(t'; y, n) = \langle \delta(y - t'_N) \rangle .$$

In Laplace space, where s is conjugate to t' and u to y,

$$\begin{aligned}
\mathop{\mathcal{L}}_{t',y} f_{t'_N,N}(t'; y, n) = \hat{f}_{t'_N,N}(s; u, n) &= \left\langle \mathrm{e}^{-ut'_n} \int_{t'_n}^{t'_{n+1}} \mathrm{d}t' \mathrm{e}^{-st'} \right\rangle \\
&= \left\langle \mathrm{e}^{-ut'_n} \mathrm{e}^{-st'_n} \frac{1 - \mathrm{e}^{-s\tau'_{n+1}}}{s} \right\rangle \\
&= \hat{\rho}(s+u)^n \frac{1 - \hat{\rho}(s)}{s} \qquad (n \geqslant 0).
\end{aligned} \tag{4.1}$$

Note that setting $u = 0$ in (4.1) gives the distribution of $N_{t'}$. We finally obtain

$$\begin{aligned}
\mathop{\mathcal{L}}_{t',y} f_{t'_N}(t'; y) &= \mathop{\mathcal{L}}_{t'} \langle \mathrm{e}^{-ut'_N} \rangle = \hat{f}_{t'_N}(s; u) \\
&= \sum_{n=0}^{\infty} \hat{f}_{t'_N,N}(s; u, n) = \frac{1}{1 - \hat{\rho}(s+u)} \frac{1 - \hat{\rho}(s)}{s}.
\end{aligned}$$

In the long-time regime, i.e. for s and u simultaneously small, we obtain the scaling form

$$\hat{f}_{t'_N}(s; u) \approx \frac{1}{\sqrt{s(s+u)}}$$

which yields

$$f_{t'_N}(t'; y) \mathop{\approx}_{t' \to \infty} \frac{1}{\pi \sqrt{y(t'-y)}}.$$

Consequently, the random variable $H' = \lim_{t' \to \infty} t'^{-1} t'_N$ possesses the limiting distribution

$$f_{H'}(x) = \frac{1}{\pi \sqrt{x(1-x)}} \tag{4.2}$$

which is the arcsine law on [0, 1].

Using the equality $t_N/t = \left(t'_N/t'\right)^{1/2\alpha}$, this last result yields immediately the distribution of

$$H = \lim_{t \to \infty} t_N/t = (H')^{1/2\alpha}$$

which reads

$$f_H(x) = \frac{2\alpha x^{\alpha-1}}{\pi \sqrt{1 - x^{2\alpha}}} = \frac{2\alpha}{\pi x \sqrt{x^{-2\alpha} - 1}}. \tag{4.3}$$

This is the main result of this section. Let us define

$$h(s, \alpha) = \langle H^s \rangle = \frac{1}{\pi} B\left(\frac{s}{2\alpha} + \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s}{2\alpha} + \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2\alpha} + 1\right)} \tag{4.4}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. For integer values of s, equation (4.4) gives the moments of f_H, denoted by

$$h_k^{(\alpha)} = h(k, \alpha) = \langle H^k \rangle .$$

In the particular case $\alpha = \frac{1}{2}$, corresponding to Brownian motion, the distribution of $H \equiv H'$ is the arcsine law (4.2), with moments

$$h_k^{(1/2)} = \frac{1}{\pi} B\left(k + \tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{(2k-1)!!}{2^k k!} = \frac{(2k)!}{2^{2k}(k!)^2}. \tag{4.5}$$

5. Determination of the moments

In order to obtain recursion relations for the moments of the random variable X, we proceed in two steps. We first compute the moments of t'_{N-1}/t'_N, from which we deduce those of t_{N-1}/t_N. The recursion relations for the $\langle X^k \rangle$ then emerge from (2.4). Equations (3.2) and (3.3) finally determine the moments of M.

5.1. Moments of t_{N-1}/t_N

We first determine the probability density function of the joint variables t'_{N-1} and t'_N. In Laplace space, we have

$$\hat{f}_{t'_{N-1},t'_N,N}(s; u, v, n) = \left\langle \mathrm{e}^{-ut'_{N-1}} \mathrm{e}^{-vt'_N} \int_{t'_n}^{t'_{n+1}} \mathrm{d}t'\, \mathrm{e}^{-st'} \right\rangle$$

$$= \begin{cases} \hat{\rho}(s+u+v)^{n-1} \hat{\rho}(s+v) \dfrac{1-\hat{\rho}(s)}{s} & (n \geqslant 1) \\ \dfrac{1-\hat{\rho}(s)}{s} & (n = 0). \end{cases}$$

Summing over n gives

$$\mathop{\mathcal{L}}_{t'} \left\langle \mathrm{e}^{-ut'_{N-1}} \mathrm{e}^{-vt'_N} \right\rangle = \hat{f}_{t'_{N-1},t'_N}(s; u, v) = \sum_{n=0}^{\infty} \hat{f}_{t'_{N-1},t'_N,N}(s; u, v, n)$$

$$= \frac{1-\hat{\rho}(s)}{s} \left(1 + \frac{\hat{\rho}(s+v)}{1-\hat{\rho}(s+u+v)} \right) \tag{5.1}$$

so that, in particular, $\hat{f}_{t'_{N-1},t'_N}(s; u = 0, v = 0) = 1/s$.

The first moment of the random variable t'_{N-1}/t'_N is obtained by considering

$$\mathop{\mathcal{L}}_{t'} \left\langle \frac{t'_{N-1}}{t'_N} \right\rangle = \int_0^{\infty} \mathrm{d}v \left(-\frac{\mathrm{d}}{\mathrm{d}u} \right)_{u=0} \mathop{\mathcal{L}}_{t'} \left\langle \mathrm{e}^{-ut'_{N-1}} \mathrm{e}^{-vt'_N} \right\rangle$$

$$= \frac{\hat{\rho}(s)}{s} + \frac{1-\hat{\rho}(s)}{s} \ln(1-\hat{\rho}(s))$$

$$\mathop{\approx}_{s\to 0} \frac{1}{s} + \frac{\ln s}{2\sqrt{s}}$$

which leads to

$$\left\langle \frac{t'_{N-1}}{t'_N} \right\rangle \mathop{\approx}_{t'\to\infty} 1 - \frac{\ln t'}{2\sqrt{\pi t'}} \tag{5.2}$$

omitting the finite parts of the logarithms.

This computation generalizes to higher-order moments, using the asymptotic form (2.1) in (5.1). We have

$$\mathop{\mathcal{L}}_{t'}\left\langle\left(\frac{t'_{N-1}}{t'_N}\right)^k\right\rangle = \left(\int_0^\infty \mathrm{d}v\right)^k\left(-\frac{\mathrm{d}}{\mathrm{d}u}\right)^k_{u=0}\mathop{\mathcal{L}}_{t'}\left\langle \mathrm{e}^{-ut'_{N-1}}\mathrm{e}^{-vt'_N}\right\rangle$$
$$\mathop{\approx}_{s\to 0}\frac{1}{s}+kh_k^{(1/2)}\frac{\ln s}{\sqrt{s}}$$

which leads to

$$\left\langle\left(\frac{t'_{N-1}}{t'_N}\right)^k\right\rangle \mathop{\approx}_{t'\to\infty} 1-kh_k^{(1/2)}\frac{\ln t'}{\sqrt{\pi t'}} \tag{5.3}$$

where $h_k^{(1/2)}$ is given by equation (4.5). In particular, since $h_1^{(1/2)} = \frac{1}{2}$, equation (5.2) is recovered.

The result (5.3) can be extended to non-integer values of k. We thus have (see (3.4))

$$\langle \ell'_N\rangle = -\left\langle \ln\frac{t'_{N-1}}{t'_N}\right\rangle = \lim_{k\to 0}\left\langle\frac{1-\left(t'_{N-1}/t'_N\right)^k}{k}\right\rangle \mathop{\approx}_{t'\to\infty}\frac{\ln t'}{\sqrt{\pi t'}}$$

as $\lim_{k\to 0} h_k^{(1/2)} = 1$. Equation (5.3) can thus be rewritten as

$$\left\langle\left(\frac{t'_{N-1}}{t'_N}\right)^k\right\rangle \mathop{\approx}_{t'\to\infty} 1-kh_k^{(1/2)}\langle\ell'_N\rangle. \tag{5.4}$$

As announced above, when $t\to\infty$, the random variable t'_{N-1}/t'_N converges to 1, in law.

The moments $f_{k,t}$ of t_{N-1}/t_N are obtained from (5.4) as

$$f_{k,t} = \left\langle\left(\frac{t_{N-1}}{t_N}\right)^k\right\rangle = \left\langle\left(\frac{t'_{N-1}}{t'_N}\right)^{k/2\alpha}\right\rangle \mathop{\approx}_{t'\to\infty} 1-\frac{k}{2\alpha}h_k^{(\alpha)}\langle\ell'_N\rangle$$

because $h(k/2\alpha, \frac{1}{2}) = h(k,\alpha)\equiv h_k^{(\alpha)}$. On the other hand,

$$\bar{\ell}_t = \langle\ell_N\rangle = -\left\langle\ln\frac{t_{N-1}}{t_N}\right\rangle = \frac{1}{2\alpha}\langle\ell'_N\rangle \mathop{\approx}_{t\to\infty}\frac{\ln t}{\sqrt{\pi}\,t^\alpha}$$

hence finally

$$f_{k,t}\mathop{\approx}_{t\to\infty} 1-kh_k^{(\alpha)}\bar{\ell}_t. \tag{5.5}$$

5.2. *Moments of X*

From the recursion relation (2.4), we have

$$\langle X_N^k\rangle = \left\langle\left(1-\frac{t_{N-1}}{t_N}X_{N-1}\right)^k\right\rangle. \tag{5.6}$$

In the long-time regime, there is an asymptotic decoupling of the variables X_{N-1} and t_{N-1}/t_N, so that it is legitimate to take $X_N\to X$, while keeping the leading time dependence of $f_{k,t}$, given by (5.5). This procedure can be justified along the lines of [11]. Consider first the simple situation $\alpha = \frac{1}{2}$. The difference between unity and t_{N-1}/t_N, which gives rise to the result (5.5), is proportional to the interval $\tau_N = t_N - t_{N-1}$. This quantity has been shown in [11] to be, asymptotically for large t, independent of t_N, and distributed according to the *a priori* law $\rho(\tau)$. A similar decoupling takes place asymptotically for generic α.

Denoting the moments $\langle X^k \rangle$ by x_k, we obtain

$$x_k = \mathcal{B}(f_{k,t} x_k) \tag{5.7}$$

with $x_0 = f_{0,t} = 1$, and where we have introduced the notation $\mathcal{B}$ for the linear binomial operator

$$\mathcal{B}(x_k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j x_j. \tag{5.8}$$

As shown in the appendix, equation (5.7) implies the following recursion relations, according to the parity of k:

$$x_k(1 + f_{k,t}) = \mathcal{B}(x_k(1 + f_{k,t})) \qquad (k \text{ odd}) \tag{5.9}$$

$$x_k(1 - f_{k,t}) = -\mathcal{B}(x_k(1 - f_{k,t})) \qquad (k \text{ even}). \tag{5.10}$$

Using the expression (5.5) of $f_{k,t}$, we obtain, in the limit $t \to \infty$, where $\bar{\ell}_t \to 0$,

$$x_k = \mathcal{B}(x_k) \qquad (k \text{ odd}) \tag{5.11}$$

$$k h_k^{(\alpha)} x_k = -\mathcal{B}(k h_k^{(\alpha)} x_k) \qquad (k \text{ even}). \tag{5.12}$$

These relations, which can be rewritten as

$$x_k = \frac{1}{2} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \qquad (k \text{ odd}) \tag{5.13}$$

$$x_k = -\frac{1}{2k h_k^{(\alpha)}} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j j h_j^{(\alpha)} x_j \qquad (k \text{ even}) \tag{5.14}$$

determine the stationary values of the x_k recursively. We thus obtain

$$x_1 = \tfrac{1}{2} \qquad x_2 = \frac{h_1^{(\alpha)}}{4h_2^{(\alpha)}} \qquad x_3 = -\frac{1}{4} + \frac{3h_1^{(\alpha)}}{8h_2^{(\alpha)}}$$

$$x_4 = -\frac{h_1^{(\alpha)} + 3h_3^{(\alpha)}}{8h_4^{(\alpha)}} + \frac{9h_1^{(\alpha)} h_3^{(\alpha)}}{16 h_2^{(\alpha)} h_4^{(\alpha)}} \cdots.$$

5.3. Moments of M

For reasons similar to those exposed below equation (5.6), the random variables H and X (defined in the limit $t \to \infty$) are independent. Thus, by (3.2) we have

$$\langle \xi^k \rangle = h_k^{(\alpha)} x_k \tag{5.15}$$

which, together with equation (3.1), leads to a determination of the even moments of the mean magnetization M in terms of the x_k:

$$\langle M^k \rangle = \langle (1 - 2\xi)^k \rangle = \mathcal{B}(2^k h_k^{(\alpha)} x_k) \qquad (k \text{ even}). \tag{5.16}$$

Thus, finally we obtain

$$
\begin{aligned}
\langle M^2\rangle &= 1 - h_1^{(\alpha)} \\
\langle M^4\rangle &= 1 + 2h_3^{(\alpha)} - \frac{3h_1^{(\alpha)}h_3^{(\alpha)}}{h_2^{(\alpha)}} \\
\langle M^6\rangle &= 1 - 5h_1^{(\alpha)} - 10h_3^{(\alpha)} - 16h_5^{(\alpha)} + \frac{h_1^{(\alpha)}(15h_3^{(\alpha)} + 20h_5^{(\alpha)})}{h_2^{(\alpha)}} \\
&\quad + \frac{(10h_1^{(\alpha)} + 30h_3^{(\alpha)})h_5^{(\alpha)}}{h_4^{(\alpha)}} - \frac{45h_1^{(\alpha)}h_3^{(\alpha)}h_5^{(\alpha)}}{h_2^{(\alpha)}h_4^{(\alpha)}}
\end{aligned}
\tag{5.17}
$$

and so on.

For instance, if $\alpha = \frac{1}{2}$, corresponding to Brownian motion, the successive even moments of M are equal to $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \ldots$, i.e.

$$\langle M^{2j}\rangle = \frac{(2j)!}{2^{2j}(j!)^2} = h_j^{(1/2)}$$

which are the even moments of the arcsine law on $[-1, 1]$ (see (7.2) below).

6. An integral equation for the determination of f_M

The recursion relation (5.11) expresses a symmetry property of the distribution f_X:

$$f_X(x) = f_X(1-x) \tag{6.1}$$

(see the appendix). This is also obvious from (3.1), since formally $F = 1$ in the present case.

The recursion relation (5.12), which can be rewritten as

$$k\langle\xi^k\rangle = -\mathcal{B}\left(k\langle\xi^k\rangle\right) \qquad (k \text{ even}) \tag{6.2}$$

expresses a symmetry property of the distribution f_ξ, as we now show. First, it is easy to prove that

$$\mathcal{B}\left(k\langle\xi^k\rangle\right) = -k\langle\xi(1-\xi)^{k-1}\rangle.$$

Therefore, equation (6.2) yields

$$\langle\xi^k\rangle = \langle\xi(1-\xi)^{k-1}\rangle \qquad (k \text{ even})$$

which is equivalent to the following symmetry property:

$$\xi f_\xi(\xi) = (1-\xi) f_\xi(1-\xi) \tag{6.3}$$

or

$$\phi(\xi) = \phi(1-\xi) \tag{6.4}$$

introducing the function

$$\phi(\xi) = \xi f_\xi(\xi). \tag{6.5}$$

On the other hand, as a consequence of (3.2) and of the independence of H and X, the distribution f_ξ is equal to the convolution of f_H, given by (4.3), and of f_X:

$$f_\xi(\xi) = \int_\xi^1 \frac{\mathrm{d}x}{x} f_X(x) f_H\left(\frac{\xi}{x}\right) = \frac{2\alpha}{\pi}\xi^{\alpha-1}\int_\xi^1 \mathrm{d}x\, \frac{f_X(x)}{\sqrt{x^{2\alpha} - \xi^{2\alpha}}} \tag{6.6}$$

hence

$$\phi(\xi) = \frac{2\alpha}{\pi}\xi^{\alpha}\int_{\xi}^{1} \mathrm{d}x \frac{f_X(x)}{\sqrt{x^{2\alpha} - \xi^{2\alpha}}}. \tag{6.7}$$

In summary, two conditions determine the distribution $f_X(x)$: it obeys the symmetry property (6.1), and the function $\phi(\xi)$, given by (6.7), obeys the symmetry property (6.4).

Once the probability density f_X is known, f_ξ is given by (6.6). Finally, equations (3.3), (6.5) and (6.4) imply

$$f_M(m) = \frac{1}{1-m^2}\phi\left(\frac{1 \pm m}{2}\right). \tag{6.8}$$

We explore the consequences of this general set-up in the next three sections.

7. The case $\alpha = \frac{1}{2}$

This situation corresponds to Brownian motion. It is easy to check that the uniform distribution on [0, 1],

$$f_X(x) = 1 \tag{7.1}$$

solves the problem. Indeed, equations (6.6) and (6.7) yield

$$\phi(\xi) = \frac{2}{\pi}\sqrt{\xi(1-\xi)} \qquad f_\xi(\xi) = \frac{2}{\pi}\sqrt{\frac{1-\xi}{\xi}}$$

which satisfy (6.3) and (6.4). Finally, by (6.8), the limiting distribution of M_t is obtained:

$$f_M(m) = \frac{1}{\pi\sqrt{1-m^2}} \tag{7.2}$$

which is the arcsine law on $[-1, 1]$.

All of these results can be derived by more direct means, using the fact that in the present case the time intervals $\tau_1, \tau_2, \ldots$ between sign changes define a renewal process [11].

8. Local analysis in the persistence region

The persistence region is defined by the condition $M \to \pm 1$, i.e. $\xi \to 0$ or $\xi \to 1$.

Considering (6.6) for $\xi \to 0$ yields at once

$$f_\xi(\xi) \underset{\xi\to 0}{\approx} \frac{2\alpha}{\pi}\left\langle X^{-\alpha}\right\rangle \xi^{\alpha-1} \tag{8.1}$$

provided the average $\left\langle X^{-\alpha}\right\rangle$ is convergent (see the comment below equation (9.12)). Consequently, using (6.5) and (6.8), we obtain

$$f_M(m) \underset{m\to\pm 1}{\approx} C(1-m^2)^{\alpha-1} \tag{8.2}$$

with

$$C = \frac{2^{1-2\alpha}\alpha}{\pi}\left\langle X^{-\alpha}\right\rangle. \tag{8.3}$$

The behaviour of the distribution $f_X(x)$ as $x \to 0$ can be determined as well. Assuming $f_X(x) \approx Ax^\gamma$ $(x \to 0)$, and using (6.1), we obtain

$$\phi(\xi) \underset{\xi\to 1}{\approx} A\frac{\sqrt{2\alpha}}{\pi}\int_0^{\bar{\xi}} \mathrm{d}\bar{x}\, \frac{\bar{x}^\gamma}{\sqrt{\bar{\xi}-\bar{x}}} = A\sqrt{\frac{2\alpha}{\pi}}\frac{\Gamma(\gamma+1)}{\Gamma\left(\gamma+\frac{3}{2}\right)}\bar{\xi}^{\gamma+1/2}$$

with $\bar{\xi} = 1-\xi$, $\bar{x} = 1-x$. An identification with (8.1), using again (6.5) and (6.3), yields the values of γ and A, hence

$$f_X(x) \underset{x\to 0}{\approx} \sqrt{\frac{2\alpha}{\pi}}\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\left\langle X^{-\alpha}\right\rangle x^{\alpha-1/2}. \tag{8.4}$$

Let us compare the singular behaviour (8.2) of f_M in the persistence region with the beta law on $[-1, 1]$ of the same index:

$$f_M^{\mathrm{beta}}(m) = C^{\mathrm{beta}}(1-m^2)^{\alpha-1} \tag{8.5}$$

where

$$C^{\mathrm{beta}} = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\alpha)}. \tag{8.6}$$

A measure of the difference between the two distributions is provided by the enhancement factor

$$E = \frac{C}{C^{\mathrm{beta}}}. \tag{8.7}$$

For $\alpha = \frac{1}{2}$, the distribution $f_M(m)$ is the arcsine law (7.2), which is a beta law. Equation (7.1) yields $\left\langle X^{-1/2}\right\rangle = 2$, so that $C = C^{\mathrm{beta}} = 1/\pi$, and $E = 1$. The estimate (8.4) also agrees with (7.1).

For $\alpha \neq \frac{1}{2}$, the distribution $f_M(m)$ is no longer a beta law, so that the enhancement factor E is non-trivial.

9. Asymptotic analysis for large values of α

For large values of α, the distributions $f_X(x)$, $f_\xi(\xi)$ and $f_M(m)$ are expected to share, at least qualitatively, some resemblance with the beta law (8.5). This observation suggests setting

$$f_X(x) \underset{\alpha\gg 1}{\sim} \exp\bigl(-\alpha S(x)\bigr) \tag{9.1}$$

with

$$S(x) = S(1-x) \tag{9.2}$$

as a consequence of (6.1). The function $S(x)$ is expected to be regular, and positive, with a minimum at $S(\frac{1}{2}) = 0$, just as its counterpart

$$S^{\mathrm{beta}}(x) = -\ln(4x(1-x)) \tag{9.3}$$

associated with the beta law (8.5).

With these hypotheses, $\phi(\xi)$, given by (6.7), can be estimated as follows. Setting $x = \xi+\varepsilon$ with $\varepsilon \ll 1$, we have $f_X(x) \approx \mathrm{e}^{-\alpha S(\xi)-\alpha\varepsilon S'(\xi)}$ and $x^{2\alpha} - \xi^{2\alpha} \approx \xi^{2\alpha}(\mathrm{e}^{2\alpha\varepsilon/\xi} - 1)$. The change of integration variable from x to $z = 2\alpha\varepsilon/\xi$ yields

$$\phi(\xi) \underset{\alpha\gg 1}{\approx} P(\xi) f_X(\xi) \tag{9.4}$$

with

$$P(\xi) = \frac{\xi}{\pi} \int_0^\infty \mathrm{d}z \, \frac{\mathrm{e}^{-\xi S'(\xi) z/2}}{\sqrt{\mathrm{e}^z - 1}}.$$

Setting $y = \mathrm{e}^{-z}$, and returning to the variable x, we finally obtain

$$P(x) = \frac{x}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2} x S'(x)\right)}{\Gamma\left(1 + \frac{1}{2} x S'(x)\right)} \tag{9.5}$$

provided the arguments of both Gamma functions are positive (this will indeed be the case). Note that (9.5) no longer involves the parameter α.

Omitting again pre-exponential factors, equation (9.4) implies

$$f_M(m) \underset{\alpha \gg 1}{\sim} \exp\left(-\alpha S\left(\frac{1 \pm m}{2}\right)\right). \tag{9.6}$$

In the regime of large α, the three distributions of interest are therefore given by a single function $S(x)$. The problem then amounts to finding $S(x)$, with the symmetry property (9.2), and such that the corresponding function $P(x)$, given by (9.5), obeys

$$P(x) = P(1 - x) \tag{9.7}$$

as a consequence of (6.4). The function $S(x)$ is entirely determined by the above conditions. This property is more evident in the present regime than in the general case of section 6, because (9.5) is explicit, while (6.7) is an integral relationship.

Let us first investigate the behaviour of $S(x)$ for $x \to \frac{1}{2}$, i.e. $m \to 0$, corresponding to the centre of the distributions. Inserting the expansion

$$S(x) = c_2 \left(x - \tfrac{1}{2}\right)^2 + c_4 \left(x - \tfrac{1}{2}\right)^4 + \cdots$$

in (9.5), (9.7), and expanding the Gamma functions accordingly, we obtain

$$c_2 = \frac{2}{\ln 2} \qquad c_4 = \frac{4}{3 \ln 2} + \frac{\pi^2}{3(\ln 2)^3} - \frac{\zeta(3)}{(\ln 2)^4} \cdots \tag{9.8}$$

and

$$P(x) = \tfrac{1}{2} + \left(\frac{\pi^2}{12(\ln 2)^2} - 3\right) \left(x - \tfrac{1}{2}\right)^2 + \cdots.$$

To leading order, keeping only the quadratic term in $S(x)$, we find that the bulk of the distributions are given asymptotically by narrow Gaussians for α large, namely

$$f_X(x) \underset{\alpha \gg 1}{\sim} f_\xi(x) \underset{\alpha \gg 1}{\sim} \exp\left(-\frac{2\alpha}{\ln 2}\left(x - \tfrac{1}{2}\right)^2\right) \qquad f_M(m) \underset{\alpha \gg 1}{\sim} \exp\left(-\frac{\alpha}{2 \ln 2} m^2\right).$$

The latter result is in agreement with the expressions (5.17) of the moments of M, which behave as $\langle M^2 \rangle \approx (\ln 2)/\alpha$, $\langle M^4 \rangle \approx 3(\ln 2)^2/\alpha^2$, and so on, for $\alpha \gg 1$.

It is also worthwhile noticing that the beta law (8.5) and (9.3) also becomes a narrow Gaussian for α large. We have $S^{\mathrm{beta}}(x) \approx 4(x - \frac{1}{2})^2$ and $f_M^{\mathrm{beta}}(m) \approx \mathrm{e}^{-\alpha m^2}$, so that the beta law misses a finite factor $2 \ln 2 \approx 1.3862$ in the variance of the mean magnetization.

The expression (9.8) of the subleading amplitude c_4, involving Riemann's zeta function, shows, however, that the function $S(x)$ is altogether non-trivial.

Let us now turn to the behaviour of $S(x)$ deep in the tails of the distributions, i.e. for $x \to 0$ or 1, or $m \to \pm 1$, corresponding to the persistence region. The general result (8.2)

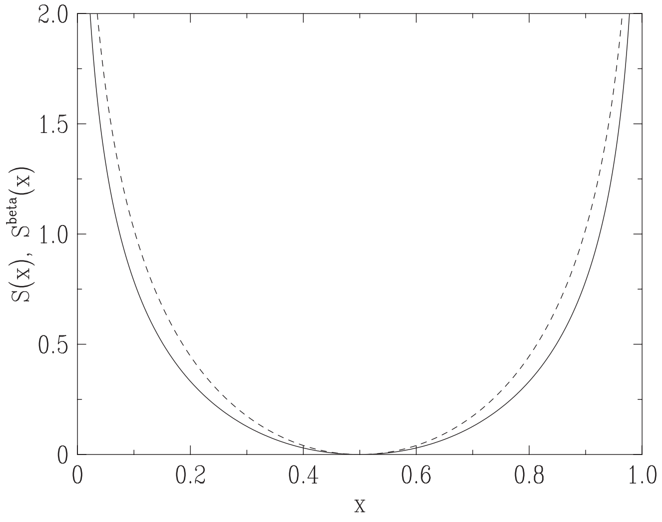

Figure 1. Plot of the function $S(x)$ characterizing the limiting distributions of the variables X and ξ and of the mean magnetization M in the large-α region, against x (full curve), compared with the function $S^{\text{beta}}(x)$ associated with the beta law (broken curve).

shows that $S(x) \approx -\ln x$ has a logarithmic divergence as $x \to 0$. Consequently, the Gamma function in the numerator of expression (9.5) for $P(x)$ becomes singular, as its argument goes to zero. Furthermore, in the same expression for $P(1-x)$, the arguments of both Gamma functions tend to infinity. A careful treatment of (9.5) yields the more complete expansions as $x \to 0$,

$$S(x) = -\ln x + S_0 + 2\sqrt{\frac{2x}{\pi}} + \cdots \qquad P(x) = \sqrt{\frac{2x}{\pi}} + \cdots \tag{9.9}$$

while the constant S_0 cannot be predicted by this local analysis. The square-root behaviour of $P(x)$ and its prefactor agree with the general results (8.1) and (8.4).

We have determined numerically the solution of (9.5) and (9.7) over the whole range $0 < x < \frac{1}{2}$, thus obtaining accurate values of $S(x)$. This approach yields, in particular, $S_0 \approx -2.0410$. Figure 1 shows a plot of the function $S(x)$ thus obtained, compared with $S^{\text{beta}}(x)$.

As the amplitude C^{beta} of the beta law (8.5) remains of the order of unity, within exponential accuracy, the result (9.9) for $S(x)$ implies that the amplitude C of the power law (8.2) in the persistence region, and the enhancement factor E defined in (8.7), blow up exponentially, as

$$C \underset{\alpha\gg 1}{\sim} E \underset{\alpha\gg 1}{\sim} \exp(G\alpha) \tag{9.10}$$

with

$$G = \lim_{x\to 0}(S^{\text{beta}}(x) - S(x)) = -S_0 - 2\ln 2 \approx 0.6547. \tag{9.11}$$

In order to test the relevance of this large-α approach, we have evaluated E numerically for various values of the parameter α, and compared the results with the exponential law (9.10) predicted for large α. The computation of E can be done in (at least) two different ways.

The first method consists in directly evaluating the limit

$$E = \lim_{n\to\infty} \frac{\langle M^{2n}\rangle}{\langle M^{2n}\rangle^{\text{beta}}}.$$

The moments of the beta law (8.5) read

$$\left\langle M^{2n}\right\rangle^{\text{beta}} = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(n+\alpha+\frac{1}{2}\right)}$$

while the true moments $\left\langle M^{2n}\right\rangle$ are determined from (5.13), (5.14) and (5.16), up to some maximal order, typically $n_{\max} = 100$–150, beyond which the numerical accuracy rapidly deteriorates, because the computation of $\left\langle M^{2n}\right\rangle$ involves alternating sums.

The second method consists in combining (8.3) and (8.6), thus giving

$$E = \frac{2^{1-2\alpha}}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\left\langle X^{-\alpha}\right\rangle$$

and in evaluating $\left\langle X^{-\alpha}\right\rangle$ as

$$\left\langle X^{-\alpha}\right\rangle = \left\langle (1-X)^{-\alpha}\right\rangle = \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!}x_n. \tag{9.12}$$

The behaviour (8.4) implies that the x_n decay as $n^{-\alpha-1/2}$, so that the term of order n in the above sum decays as $n^{-3/2}$. Hence this sum is convergent, and truncating it at some order $n_{\max}$ brings a correction proportional to $n_{\max}^{-1/2}$. The x_n are again determined from (5.13) and (5.14). A linear extrapolation in $n_{\max}^{-1/2}$ of the results of both schemes turns out to yield consistent results. We have, for instance, $E \approx 1.443$ for $\alpha = 1$.

Figure 2 shows our numerical results for the enhancement factor E, for values of α up to 3. The comparison with the exponential law (9.10) is convincing, in spite of the moderate values of α used.

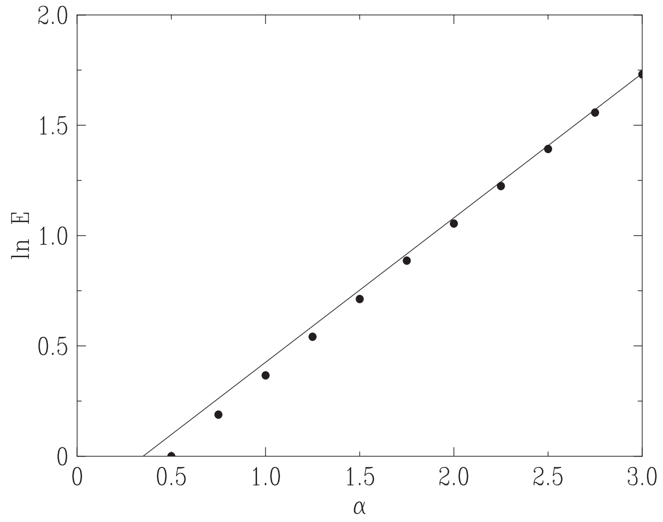

Figure 2. Logarithmic plot of the enhancement factor E in the persistence region, against α, evaluated numerically as described in the text (symbols). The straight line has the theoretical slope G of (9.11).

10. Revisiting the work of Dhar and Majumdar

10.1. Using the method of section 3.2

Let us first show that the expressions (5.17) for the even moments of M are the same as those obtained by using (3.9), with the expression of $\hat{A}(s)$ appropriate to the process (1.1), as done in [4].

The autocorrelation $A(|\Delta T|) = \langle \sigma_T \sigma_{T+\Delta T} \rangle$ of the sign process σ_t in the logarithmic time scale $T = \ln t$ reads $A(T) = (2/\pi) \arcsin(\mathrm{e}^{-\alpha|T|})$ [4], with the Laplace transform

$$\hat{A}(s) = \frac{1}{s} \left[1 - \frac{1}{\pi} B \left(\frac{s}{2\alpha} + \frac{1}{2}, \frac{1}{2} \right) \right]. \tag{10.1}$$

We note that $\hat{A}(s)$ is related to $h(s, \alpha)$, defined in (4.4), by

$$h(s, \alpha) = 1 - s\hat{A}(s) \tag{10.2}$$

where the right-hand side is equal to $2g(s)/s$ by (3.6). Using this identity, it is easy to check that the moments (3.9) obtained by the method of section 3.2, with $\hat{A}_k$ given by (10.1) for $s = k$ integer, are identical to the moments (5.17) obtained by the method of this paper.

It is, however, not possible to identify the intermediate results of both methods, as can be seen by comparing, respectively, equation (5.5) to equations (3.7) and (3.5), and equation (4.4) to equation (3.8). This demonstrates the formal character of the application of the method of section 3.2 to the process (1.1). (See also the discussion in section 3.3.)

10.2. Comments on the results obtained using Kac's formalism

A first comment is that the recursion relations for the coefficients c_k appearing in equations (14) of [4] can be easily recognized to be identical to the recursion relations (5.13) and (5.14) for x_k, by noting the correspondences

$$\begin{aligned} c_k &= \frac{2^k}{k!D_{-k/\alpha}(0)} x_k \\ \frac{D_{-k/\alpha+1}(0)}{D_{-k/\alpha}(0)} &= \frac{\sqrt{2\pi}}{\alpha} \frac{k h_k^{(\alpha)}}{2}. \end{aligned} \tag{10.3}$$

The second comment concerns the continuity conditions expressed in equations (13) of [4]. Using (10.3), these conditions yield, with the notation of this paper,

$$\begin{aligned} &\left\langle \mathrm{e}^{a(2X-1)} \right\rangle = \left\langle \mathrm{e}^{a(1-2X)} \right\rangle \\ &\left\langle \xi \, \mathrm{e}^{a(2\xi-1)} \right\rangle = \left\langle \xi \, \mathrm{e}^{a(1-2\xi)} \right\rangle. \end{aligned}$$

These equations hold for a arbitrary, hence they are equivalent to (6.1) and (6.3), respectively.

11. Summary and discussion

In this paper we have revisited and extended the work of Dhar and Majumdar [4]. Besides providing a new recursive determination of the moments of the mean magnetization M, the present study leads to a functional integral equation for the distribution of the latter quantity. This framework allows a local analysis of this distribution, and of other relevant quantities, in the persistence region ($M_t \to \pm 1$), as well as a detailed investigation of the regime where α is large.

This paper casts new light on the status of the expressions (3.9) for the moments of M. The method recalled in section 3.2, which leads to these relations, can be applied to any smooth process for which the intervals of time ℓ_N between sign changes are independent, on a logarithmic scale. For this class of processes $\langle \ell_N \rangle = \bar{\ell}$ is finite (i.e. non-zero), and the mean number of sign changes between 0 and t scales as $\langle N_t \rangle \approx (\ln t)/\bar{\ell}$.

Relations (3.9) are also verified for the class of processes considered in this work. This was observed in [4] (by comparing the expressions thus found with those obtained by another method, based on a formalism due to Kac), and justified by the absence of $\bar{\ell}$ in equations (3.9). Yet, as discussed in section 3.3, in the present case there is no obvious reason to work with a logarithmic time scale, since $\langle \ell_N \rangle$ vanishes asymptotically, and the mean number of sign changes scales as $\langle N_t \rangle \approx 2\pi^{-1/2} t^{\alpha}$ [11]. (See also the discussion in section 10.1.)

Most of the effort of this paper was to provide a new derivation of (3.9) for the class of processes (1.1). Our approach relies on the fact that the time intervals τ'_n between two sign changes of the process (1.1) form a renewal process (the τ'_n are independent, identically distributed random variables). The derivation proceeds in two steps. First, relations (5.17) for the $\langle M^k \rangle$ are obtained; then, using (10.2), equations (5.17) yield (3.9). This extends the range of applicability of relations (3.9). Note that for diffusion (in the independent-interval approximation) (see section 3.2), relations (3.9) hold but neither (5.17) nor (10.2) do.

We conclude by making a few additional comments.

In passing, let us mention another equivalent formulation of (10.2), namely that the two-time autocorrelation of the sign process reads, with $t < t'$,

$$C(t, t') = \int_0^{t/t'} \mathrm{d}x\, f_H(x). \tag{11.1}$$

Another situation where (3.9), (5.17) and (10.2) or (11.1) hold is for the renewal processes considered in [11] (provided $\theta < 1$), which are yet another deformation of Brownian motion.

Note that the first relation of (3.9), $\langle M^2 \rangle = \hat{A}_1$, holds whenever the two-time autocorrelation function is a scaling function of the ratio of the two times [5], while the first relation of (5.17), $\langle M^2 \rangle = 1 - \langle H \rangle$, does not hold in general. For instance, for the random acceleration problem, using results of [20], we find $\langle M^2 \rangle = 3\sqrt{3}/\pi - 1 \approx 0.653\,986$ and $1 - \langle H \rangle \approx 0.791\,335$.

This work also underlines the importance of the random variables X and H. The distribution of the latter is known exactly in the present case. This quantity, which is a natural one to consider for Brownian motion [1], and more generally for renewal processes [11], also appears in the context of phase ordering [21].

As mentioned in the introduction, the process (1.1) has been proposed [2] as a Markovian approximation to fractional Brownian motion. Let us compare the expressions of $\langle M^2 \rangle$ for these two processes. For the present model we have (see (5.17))

$$\langle M^2 \rangle = 1 - \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2\alpha} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2\alpha} + 1\right)} \tag{11.2}$$

while for fractional Brownian motion, with Hölder index $0 < h < 1$, we have

$$\langle M^2 \rangle = \frac{2}{\pi} \int_0^1 \mathrm{d}x\, \arcsin \frac{x^{2h} + 1 - (1-x)^{2h}}{2x^h}. \tag{11.3}$$

The correspondence between the two processes is made by identifying their persistence exponents: $\theta = \alpha = 1 - h$. For $\theta = \frac{1}{2}$, we have $\langle M^2 \rangle = \frac{1}{2}$ in both cases. For $\theta = 1$, (11.2) yields $\langle M^2 \rangle = 1 - 2/\pi \approx 0.363\,380$, while (11.3) yields $\langle M^2 \rangle = \frac{1}{3}$. For $\theta \to 0$, we have $\langle M^2 \rangle = 1 - c\sqrt{\theta}$, with (11.2) yielding $c = \sqrt{2/\pi} \approx 0.797\,885$, and (11.3) yielding $c \approx 0.812\,233$. The distributions of the mean magnetization for the two processes are therefore expected to be rather similar (for $0 < \theta < 1$).

Finally, let us comment on the changes in behaviour induced by letting the persistence exponent α vary, and compare the present process with other ones in this respect. The distribution of M shows a change in shape as α increases, the most probable value of the mean magnetization shifting from the edges to the centre [4]. More precisely, as shown in section 8, as long as $\alpha < 1$, $f_M(m)$ diverges at $m \to \pm 1$, while for $\alpha > 1$ it vanishes at these points (see equation (8.2)). However, for any arbitrary value of α the magnetization M remains distributed.

This behaviour is actually generic, whenever the two-time autocorrelation function of the process is asymptotically a function of the ratio of the two-time variables [5]. In particular, this is so for diffusion. In the independent-interval approximation the persistence exponent $\theta(D) \approx 0.1454\sqrt{D}$ increases without bound when the dimension of space D is large [18, 19]. As originally noted in [5], as long as $\theta < 1$ the density $f_M(m)$ diverges at the edges, while it vanishes there if $\theta > 1$. This was also emphasized in [9], on the basis of scaling arguments, and was recently confirmed by direct numerical computations [22].

In contrast, there are other processes for which the change in behaviour at $\theta = 1$ is more radical. For fractional Brownian motion, $\theta = 1$ appears as a maximum persistence exponent. For the renewal processes considered in [11], the mean magnetization possesses a non-trivial asymptotic distribution only if $\theta < 1$.

Appendix. Properties of the binomial operator $\mathcal{B}$

The aim of this appendix is to prove the following property, used in section 5.2. Assume that the sequence x_k satisfies

$$x_k = \mathcal{B}(f_k x_k) \tag{A.1}$$

with $x_0 = f_0 = 1$, and where $\mathcal{B}(x_k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j x_j$ (see (5.8)). Then

$$x_k(1 + f_k) = \mathcal{B}(x_k(1 + f_k)) \qquad (k \text{ odd}) \tag{A.2}$$

$$x_k(1 - f_k) = -\mathcal{B}(x_k(1 - f_k)) \qquad (k \text{ even}) \tag{A.3}$$

which are, respectively, equations (5.9) and (5.10) in the text.

A.1. Basic properties

In order to prove (A.2) and (A.3) we need the following auxiliary properties.

First, $\mathcal{B}$ is its own inverse:

$$\mathcal{B} = \mathcal{B}^{-1}. \tag{A.4}$$

A combinatorial proof of this result can be found in [23]. An alternative proof is obtained by noting that the action of $\mathcal{B}$ on exponential sequences $x_k = y^k$ reads

$$\mathcal{B}(y^k) = (1 - y)^k. \tag{A.5}$$

This relation is invariant in the change of y to $1 - y$, hence (A.4) follows.

Then, we have the properties

$$x_k = \mathcal{B}(x_k) \quad \text{for } k \text{ even} \qquad \text{implies} \quad x_k = \mathcal{B}(x_k) \quad \text{for all } k \tag{A.6}$$

$$x_k = \mathcal{B}(x_k) \quad \text{for } k \text{ odd} \qquad \text{implies} \quad x_k = \mathcal{B}(x_k) \quad \text{for all } k \tag{A.7}$$

$$x_k = -\mathcal{B}(x_k) \quad \text{for } k \text{ even} \qquad \text{implies} \quad x_k = -\mathcal{B}(x_k) \quad \text{for all } k \tag{A.8}$$

$$x_k = -\mathcal{B}(x_k) \quad \text{for } k \text{ odd} \qquad \text{implies} \quad x_k = -\mathcal{B}(x_k) \quad \text{for all } k. \tag{A.9}$$

Before giving the proofs, let us make explicit the meaning of (A.6)–(A.9).

Let us take the example of (A.6). By hypothesis, the sequence x_k satisfies the condition $x_k = \mathcal{B}(x_k)$ for k even, with x_0 arbitrary, which is equivalent to saying that

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j = 0 \qquad (k \text{ even}). \tag{A.10}$$

This recursion determines x_k for k odd in terms of the x_ℓ with $\ell = 0, \dots, k-1$ even:

$$x_1 = \tfrac{1}{2}x_0 \qquad x_3 = \tfrac{3}{2}x_2 - \tfrac{1}{4}x_0 \qquad x_5 = \tfrac{5}{2}x_4 - \tfrac{5}{2}x_2 + \tfrac{1}{2}x_0 \dots .$$

The property (A.6) states that $x_k = \mathcal{B}(x_k)$ for k odd, or equivalently,

$$2x_k = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \qquad (k \text{ odd})$$

which provides an infinite number of consistency relations amongst the x_k satisfying (A.10).

Similarly, taking the example of (A.8), by hypothesis we have $x_k = -\mathcal{B}(x_k)$ for k even, with $x_0 = 0$, which is equivalent to

$$2x_k = -\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \qquad (k \text{ even}). \tag{A.11}$$

This recursion determines x_k for k even in terms of the x_ℓ with $\ell = 1, \dots, k-1$ odd:

$$x_2 = x_1 \qquad x_4 = 2x_3 - x_1 \qquad x_6 = 3x_5 - 5x_3 + 3x_1 \dots .$$

The property (A.8) states that $x_k = -\mathcal{B}(x_k)$ for k odd, or equivalently,

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j = 0 \qquad (k \text{ odd})$$

which provides an infinite number of consistency relations amongst the x_k satisfying (A.11).

We now prove the properties (A.6)–(A.9). In order to do so, let us define, for a given sequence x_k, the Laurent series

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k} \qquad G(z) = \sum_{k=0}^{\infty} \mathcal{B}(x_k) z^{-k}.$$

We assume that these series are convergent for $|z|$ larger than some radius R. This happens, for example, if the x_k are bounded.

The functions $F(z)$ and $G(z)$ are related to each other by

$$F(z) = \frac{z}{z-1} G(1-z) \tag{A.12}$$

$$G(z) = \frac{z}{z-1} F(1-z) \tag{A.13}$$

as we now show. We have

$$x_k = \oint \frac{\mathrm{d}y}{2\pi \mathrm{i} y} y^k F(y)$$

hence, using (A.5),

$$\mathcal{B}(x_k) = \oint \frac{\mathrm{d}y}{2\pi \mathrm{i} y} (1-y)^k F(y)$$

so that

$$G(z) = \oint \frac{\mathrm{d}y}{2\pi \mathrm{i} y} \frac{z}{y+z-1} F(y).$$

This integral is equal to the contribution of the pole at $y = 1 - z$, yielding (A.13), from which (A.12) follows. The symmetric form of the formulae (A.12) and (A.13) is due to the property (A.4).

Proof of (A.6) and (A.7). The hypothesis in (A.6) implies $F(z) + F(-z) = G(z) + G(-z)$, i.e.

$$\Phi(z) = -\Phi(-z) \tag{A.14}$$

with, using (A.13),

$$\Phi(z) = F(z) - G(z) = F(z) - \frac{z}{z-1} F(1-z).$$

Therefore, $(z-1)\Phi(z) + z\Phi(1-z) = 0$, which can be rewritten, using (A.14), as

$$\frac{\Phi(z)}{z} = \frac{\Phi(z-1)}{z-1}.$$

The function $\Phi(z)/z$ is thus periodic, with unit period, and decaying at infinity, as we have $\Phi(z)/z \approx (x_0 - 2x_1)/z^2$ *a priori*. We conclude that $\Phi(z) = 0$ identically, that is $F(z) = G(z)$, implying the property (A.6).

For the case where the $x_k = \langle X^k \rangle$ are the moments of a random variable X, with density f_X on [0, 1], an alternative proof of (A.6) is as follows. The hypothesis in (A.6) expresses the property

$$\langle X^k \rangle = \langle (1-X)^k \rangle \qquad (k \text{ even}).$$

As both random variables X and $1 - X$ are positive, this last condition is sufficient to imply $f_X(x) = f_X(1-x)$, hence $\langle X^k \rangle = \langle (1-X)^k \rangle$ for all k, which proves (A.6).

The proof of the second property, (A.7), is very similar. The hypothesis in (A.7) implies

$$\frac{\Phi(z)}{z} = -\frac{\Phi(z-1)}{z-1}.$$

The function $\Phi(z)/z$ is therefore periodic, with period two, and decaying at infinity, hence $\Phi(z) = 0$ identically. □

Proof of (A.8) and (A.9). The hypothesis in (A.8) implies $F(z)+F(-z) = -(G(z)+G(-z))$, i.e.

$$\Psi(z) = -\Psi(-z) \tag{A.15}$$

with, using (A.13),

$$\Psi(z) = F(z) + G(z) = F(z) + \frac{z}{z-1}F(1-z).$$

Therefore, $(z-1)\Psi(z) - z\Psi(1-z) = 0$, which can be rewritten, using (A.15), as

$$\frac{\Psi(z)}{z} = -\frac{\Psi(z-1)}{z-1}.$$

The function $\Psi(z)/z$ is thus again periodic, with period two, and decaying at infinity, hence identically zero. The proof of the fourth property, (A.9), is very similar. □

A.2. Proofs of equations (A.2) and (A.3)

Equation (A.1) implies the relations

$$x_k(1+f_k) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_j x_j \qquad (k \text{ odd}) \tag{A.16}$$

$$x_k(1-f_k) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j f_j x_j \qquad (k \text{ even}) \tag{A.17}$$

which determines the x_k recursively. We have thus

$$x_1 = \frac{1}{1+f_1} \qquad x_2 = \frac{1-f_1}{(1+f_1)(1-f_2)} \qquad \cdots. \tag{A.18}$$

Since the operator $\mathcal{B}$ is its own inverse, (A.1) is equivalent to $f_k x_k = \mathcal{B}(x_k)$, which itself implies

$$x_k(1+f_k) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \qquad (k \text{ odd}) \tag{A.19}$$

$$x_k(f_k-1) = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j \qquad (k \text{ even}). \tag{A.20}$$

Comparing (A.16) and (A.19) shows that

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j (1-f_j) = 0 \qquad (k \text{ odd})$$

hence, using the property (A.9),

$$x_k(1-f_k) = -\mathcal{B}(x_k(1-f_k)) \qquad (k \text{ even})$$

which is equation (A.3). Similarly, comparing (A.17) and (A.20) shows that

$$\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j x_j (1+f_j) = 0 \qquad (k \text{ even})$$

or, using the property (A.6),

$$x_k(1+f_k) = \mathcal{B}(x_k(1+f_k)) \qquad (k \text{ odd})$$

which is equation (A.2).

References

[1] Feller W 1968 *An Introduction to Probability Theory and its Applications* vols 1 and 2 (New York: Wiley)
[2] Sebastian K L 1995 *J. Phys. A: Math. Gen.* **28** 4305 and references therein
[3] Majumdar S N, Bray A J, Cornell S J and Sire C 1996 *Phys. Rev. Lett.* **77** 3704
[4] Dhar A and Majumdar S N 1999 *Phys. Rev.* E **59** 6413
[5] Dornic I and Godrèche C 1998 *J. Phys. A: Math. Gen.* **31** 5413
[6] Baldassarri A, Bouchaud J P, Dornic I and Godrèche C 1999 *Phys. Rev.* E **59** R20
[7] Drouffe J M and Godrèche C 1998 *J. Phys. A: Math. Gen.* **31** 9801
[8] Godrèche C 1999 *Self-Similar Systems* ed V B Priezzhev and V P Spiridonov (Dubna: Joint Institute for Nuclear Research)
[9] Newman T J and Toroczkai Z 1998 *Phys. Rev.* E **58** R2685
[10] Toroczkai Z, Newman T J and Das S Sarma 1999 *Phys. Rev.* E **60** R1115
[11] Godrèche C and Luck J M 2000 *Preprint* cond-mat/0010428
[12] Lévy P 1939 *Compos. Math.* **7** 283
[13] Kac M 1949 *Trans. Am. Math. Soc.* **65** 1
[14] Kesten H 1973 *Acta Math.* **131** 208
[15] Kesten H, Kozlov M V and Spitzer F 1975 *Compos. Math.* **30** 145
[16] Vervaat W 1979 *Adv. Appl. Probab.* **11** 50
[17] de Calan C, Luck J M, Nieuwenhuizen Th M and Petritis D 1985 *J. Phys. A: Math. Gen.* **18** 501
[18] Majumdar S N, Sire C, Bray A J and Cornell S J 1996 *Phys. Rev. Lett.* **77** 2867
[19] Derrida B, Hakim V and Zeitak R 1996 *Phys. Rev. Lett.* **77** 2871
[20] De Smedt G, Godrèche C and Luck J M 2000 *Preprint* cond-mat/0009001 (*Europhys. Lett.* at press)
[21] Frachebourg L, Krapivsky P L and Redner S 1997 *Phys. Rev.* E **55** 6684
[22] Newman T J and Loinaz W 2000 *Preprint* cond-mat/0009365
[23] Graham R L, Knuth D E and Patashnik O 1994 *Concrete Mathematics* (Reading, MA: Addison-Wesley)